Journal of Pure and Applied Mathematics: Advances and Applications
Volume 3, Number 1, 2010, Pages 123-136

ON THE FUNDAMENTAL MATRIX OF LINEAR SYSTEMS WITH VARIABLE COEFFICIENTS

KOUICHI MURAKAMI

Department of Mathematical Sciences
Faculty of Integrated Arts and Sciences
The University of Tokushima
Tokushima 770-8502
Japan
e-mail: murakami@ias.tokushima-u.ac.jp

Abstract

We give the conditions that the linear system $x' = (A(t) + B(t))x$ is converted into another solvable linear system under the transformation based on the fundamental matrix. Using our results, we show several examples of linear systems which we can obtain the fundamental matrix.

1. Introduction

Consider the linear system

$$x' = (A(t) + B(t))x, \tag{1.1}$$

where $A(t)$ and $B(t)$ are continuous real $n \times n$ matrix functions. If both $A(t)$ and $B(t)$ are constant matrices A and B respectively, then the fundamental matrix of (1.1) is given by $e^{(A+B)t}$. In particular, if A and B are commutative, then we have $e^{(A+B)t} = e^{At}e^{Bt}$. If only $A(t)$ is a

2010 Mathematics Subject Classification: 34A05, 34A30.

Keywords and phrases: fundamental matrix, decomposition.

Received February 23, 2010

constant matrix A, and A and $B(t)$ are commutative, then the fundamental matrix of (1.1) is given by $e^{At}\Psi(t)$ where $\Psi(t)$ is a fundamental matrix of $x' = B(t)x$. However, if A and $B(t)$ are not always assumed to be commutative, it is generally difficult to obtain the fundamental matrix of (1.1).

By using the transformation $x = e^{St}y$, Yamamoto [4] gives the condition that

$$x' = A(t)x \tag{1.2}$$

is converted into another linear system $y' = B(t)y$ as follows.

Lemma 1.1. *There exists a constant matrix* S *and a continuously differentiable function* $B(t)$ *such that* (1.2) *is converted into* $y' = B(t)y$ *under the transformation* $x = e^{St}y$, *if and only if* $A(t)$ *satisfies that*

$$\begin{cases} A'(t) = SA(t) - A(t)S + e^{St}B'(t)e^{-St}, \\ A(0) = S + B(0). \end{cases} \tag{1.3}$$

From this result, we can see that, if $A(t)$ satisfies (1.3) for some S and $B(t)$, then the fundamental matrix of (1.2) is given by $e^{St}\Psi(t)$ where $\Psi(t)$ is the fundamental matrix of $y' = B(t)y$. Thus, we may take it that $A(t)$ is decomposed by S and $B(t)$ in a sense, and the fundamental matrix of (1.2) is expressed by the product of e^{St} and $\Psi(t)$ which are the fundamental matrices of $x' = Sx$ and $y' = B(t)y$ respectively. Here, we note that S and $B(t)$ are not always assumed to be commutative. Unfortunately, it is not easy to find S and $B(t)$ for given $A(t)$. Moreover, even if we find such S and $B(t)$, we need the fundamental matrix of $y' = B(t)y$ in order to obtain the fundamental matrix of (1.2). Thus, except for the case that $B(t)$ is a constant matrix, we can hardly apply Lemma 1.1 to concrete examples to obtain the fundamental matrix of (1.2).

In this paper, using the above idea, we decompose the linear system as (1.1), and intend to convert it into another solvable linear system by using the transformation $x = \Phi(t)y$ where $\Phi(t)$ is the fundamental matrix of $x' = A(t)x$. If we can obtain the fundamental matrix $\Psi(t)$ of the converted linear system, then the fundamental matrix of (1.1) is expressed by $\Phi(t)\Psi(t)$. As a sufficient condition to obtain the fundamental matrix of the linear system, we use the following well-known result.

Lemma 1.2. *Assume that* $A(t)$ *satisfies that*

$$A(t)\left(\int_0^t A(s)\,ds\right) = \left(\int_0^t A(s)\,ds\right)A(t). \tag{1.4}$$

Then, the fundamental matrix of $x' = A(t)x$ *is given by* $e^{\int_0^t A(s)ds}$.

In the following, we prepare a concrete matrix form of $A(t)$ which satisfies (1.4), and gives the explicit expression of the fundamental matrix. And then, we derive the conditions that (1.1) is converted into the linear system, which satisfies (1.4). We also apply our results to several examples of linear systems.

2. Main Results

Lemma 2.1. *Assume that* $A(t)$ *is decomposed as*

$$A(t) = a(t)E + b(t)C, \tag{2.1}$$

where $a(t), b(t)$ *are continuous real functions, E is the identity matrix, and C is a constant real* $n \times n$ *matrix. Then* $A(t)$ *satisfies* (1.4). *If* $A(t)$ *is a* 2×2 *matrix function, then* (2.1) *is a necessary and sufficient condition for* $A(t)$ *to satisfy* (1.4).

Remark 2.1. In Lemma 2.1, the expression of

$$A(t) = a(t)E + b(t)C$$

is not unique. For example, if we let

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{11} \end{pmatrix} + \begin{pmatrix} 0 & c_{12} \\ c_{21} & c_{22} - c_{11} \end{pmatrix} = c_{11}E + C',$$

then we can rewrite $A(t)$ as

$$A(t) = (a(t) + c_{11}b(t))E + b(t)C'.$$

Remark 2.2. If $A(t)$ is given by (2.1), then the fundamental matrix of $x' = A(t)x$ is expressed by

$$e^{\alpha(t)E + \beta(t)C} = e^{\alpha(t)E}e^{\beta(t)C},$$

where $\alpha(t) = \int_0^t a(s)\,ds$ and $\beta(t) = \int_0^t b(s)\,ds.$

Assume that the characteristic polynomial of C is given by

$$|\lambda E - C| = \prod_{i=1}^{r}(\lambda - \lambda_i)^{n_i}.$$

We decompose $1/|\lambda E - C|$ into partial fractions as

$$\frac{1}{|\lambda E - C|} = \sum_{i=1}^{r}\frac{h_i(\lambda)}{(\lambda - \lambda_i)^{n_i}},$$

where each $h_i(\lambda)$ is at most $(n_i - 1)$-degree polynomial of λ. Multiplying both sides of this identity by $|\lambda E - C| = \prod_{i=1}^{r}(\lambda - \lambda_i)^{n_i}$, we have

$$1 = \sum_{i=1}^{r} h_i(\lambda)\prod_{\substack{j=1 \\ j \neq i}}^{r}(\lambda - \lambda_j)^{n_j}. \tag{2.2}$$

Define the projection on the generalized eigenspace associated with λ_i as

$$P_i = h_i(C)\prod_{\substack{j=1 \\ j \neq i}}^{r}(C - \lambda_j E)^{n_j}. \tag{2.3}$$

From (2.2), we have $\sum_{i=1}^{r} P_i = E$. Also, since we know that $\prod_{i=1}^{r} (C - \lambda_i E)^{n_i} = O$ by Hamilton-Cayley theorem, we have $P_i P_j = O$, for $i \neq j$. Therefore, we have

$$e^{\beta(t)C} = \sum_{i=1}^{r} e^{\beta(t)C} P_i = \sum_{i=1}^{r} e^{\beta(t)(\lambda_i E + (C - \lambda_i E))} P_i$$

$$= \sum_{i=1}^{r} e^{\lambda_i \beta(t) E} e^{\beta(t)(C - \lambda_i E)} P_i$$

$$= \sum_{i=1}^{r} e^{\lambda_i \beta(t)} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \beta(t)^k (C - \lambda_i E)^k \right) P_i$$

$$= \sum_{i=1}^{r} e^{\lambda_i \beta(t)} \left(\sum_{k=0}^{n_i - 1} \frac{1}{k!} \beta(t)^k (C - \lambda_i E)^k \right) P_i.$$

Thus, we have

$$e^{\alpha(t)E + \beta(t)C} = \sum_{i=1}^{r} e^{\alpha(t) + \lambda_i \beta(t)} \left(\sum_{k=0}^{n_i - 1} \frac{1}{k!} \beta(t)^k (C - \lambda_i E)^k \right) P_i.$$

Lemma 2.2. *Assume that* $A(t)$ *is given by* (2.1). *Then, the fundamental matrix of* $x' = A(t)x$ *is expressed by*

$$e^{\alpha(t)E + \beta(t)C} = \sum_{i=1}^{r} e^{\alpha(t) + \lambda_i \beta(t)} \left(\sum_{k=0}^{n_i - 1} \frac{1}{k!} \beta(t)^k (C - \lambda_i E)^k \right) P_i,$$

where $\alpha(t) = \int_0^t a(s)\,ds$, $\beta(t) = \int_0^t b(s)\,ds$, *each* λ_i *is the eigenvalue with the multiplicity* n_i *of* C, *and each* P_i *is defined by* (2.3). *In particular, if* $A(t)$ *is a* 2×2 *matrix function, then the fundamental matrix of* $x' = A(t)x$ *where* $A(t)$ *is given by* (2.1) *is expressed in the following.*

(i) *If C has real eigenvalues* λ_1 *and* λ_2 $(\lambda_1 \neq \lambda_2)$, *then we have*

$$e^{\alpha(t)E+\beta(t)C} = \frac{e^{\alpha(t)+\lambda_1\beta(t)}}{\lambda_1 - \lambda_2}(C - \lambda_2 E) + \frac{e^{\alpha(t)+\lambda_2\beta(t)}}{\lambda_2 - \lambda_1}(C - \lambda_1 E).$$

(ii) *If C has real double eigenvalues* λ, *then we have*

$$e^{\alpha(t)E+\beta(t)C} = e^{\alpha(t)+\lambda\beta(t)}(E + \beta(t)(C - \lambda E)).$$

(iii) *If C has complex conjugate eigenvalues* $\lambda = \mu + i\nu$ *and* $\overline{\lambda} = \mu - i\nu$, *then we have*

$$e^{\alpha(t)E+\beta(t)C} = e^{\alpha(t)+\mu\beta(t)}\left(\cos(\nu\beta(t))E + \frac{\sin(\nu\beta(t))}{\nu}(C - \mu E)\right).$$

Example 2.1. Consider the linear system

$$x' = \begin{pmatrix} -3 + t\sin t & \cos t \\ \cos t & -3 + t\sin t \end{pmatrix} x. \tag{2.4}$$

We decompose the coefficient matrix of (2.4) as

$$A(t) = (-3 + t\sin t)E + (\cos t)C, \quad \text{where} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let

$$\alpha(t) = \int_0^t (-3 + s\sin s)\,ds = -3t - t\cos t + \sin t,$$

$$\beta(t) = \int_0^t \cos s\,ds = \sin t.$$

Since the eigenvalues of C are $\lambda = \pm 1$, Lemma 2.2 implies that the fundamental matrix (2.4) is

$$\begin{aligned} e^{\alpha(t)E+\beta(t)C} &= \frac{e^{\alpha(t)+\beta(t)}}{1-(-1)}(C + E) + \frac{e^{\alpha(t)-\beta(t)}}{-1-1}(C - E) \\ &= \frac{e^{\alpha(t)+\beta(t)}}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{e^{\alpha(t)-\beta(t)}}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

$$= \frac{e^{\alpha(t)-\beta(t)}}{2}\begin{pmatrix} 1+e^{2\beta(t)} & -1+e^{2\beta(t)} \\ -1+e^{2\beta(t)} & 1+e^{2\beta(t)} \end{pmatrix}$$

$$= \frac{e^{-t(3+\cos t)}}{2}\begin{pmatrix} 1+e^{2\sin t} & -1+e^{2\sin t} \\ -1+e^{2\sin t} & 1+e^{2\sin t} \end{pmatrix}.$$

Theorem 2.1. *Consider the linear system*

$$x' = (A(t) + B(t))x, \tag{2.5}$$

where $A(t)$ is a continuous $n \times n$ matrix function, and $B(t)$ is a continuously differentiable $n \times n$ matrix function. If $B(t)$ satisfies

$$B'(t) = A(t)B(t) - B(t)A(t),$$

then (2.5) *is converted into*

$$y' = B(0)y,$$

under the transformation $x = \Phi(t)y$ where $\Phi(t)$ is the fundamental matrix of $x' = A(t)x$ satisfying $\Phi(0) = E$.

Proof. By the transformation $x = \Phi(t)y$, the left hand side of (2.5) is given by

$$x' = (\Phi(t)y)' = A(t)\Phi(t)y + \Phi(t)y'.$$

Also, the right hand side of (2.5) is given by

$$(A(t) + B(t))x = (A(t) + B(t))\Phi(t)y.$$

Therefore (2.5) is converted into

$$y' = \Phi(t)^{-1}B(t)\Phi(t)y.$$

Since $(\Phi(t)^{-1})' = -\Phi(t)^{-1}A(t)$, we have

$$(\Phi(t)^{-1}B(t)\Phi(t))'$$

$$= -\Phi(t)^{-1}A(t)B(t)\Phi(t) + \Phi(t)^{-1}B'(t)\Phi(t) + \Phi(t)^{-1}B(t)A(t)\Phi(t)$$

$$= \Phi(t)^{-1}(-A(t)B(t) + B'(t) + B(t)A(t))\,\Phi(t)$$

$$= O.$$

Thus, we have that $\Phi(t)^{-1}B(t)\Phi(t)$ is a constant matrix or $\Phi(t)^{-1}B(t)\Phi(t) \equiv B(0)$. Therefore (2.5) is converted into

$$y' = B(0)y.$$

This completes the proof. □

Example 2.2. Consider the linear system

$$x' = \begin{pmatrix} 1 & e^{2t} \\ -e^{-2t} & -1 \end{pmatrix} x. \tag{2.6}$$

We decompose (2.6) as

$$x' = (A + B(t))x,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & e^{2t} \\ -e^{-2t} & 0 \end{pmatrix}.$$

Then, the fundamental matrix of $x' = Ax$ is given by

$$\Phi(t) = e^{At} = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Also, we have

$$AB(t) - B(t)A = \begin{pmatrix} 0 & e^{2t} \\ e^{-2t} & 0 \end{pmatrix} - \begin{pmatrix} 0 & -e^{2t} \\ -e^{-2t} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2e^{2t} \\ 2e^{-2t} & 0 \end{pmatrix} = B'(t).$$

Therefore, by the transformation $x = \Phi(t)y$, (2.6) is converted into

$$y' = B(0)y, \quad \text{where} \quad B(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since the eigenvalues of $B(0)$ are $\lambda = \pm i$, we have

$$e^{B(0)t} = (\cos t)E + (\sin t)B(0) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Hence, we have the fundamental matrix of (2.6) as

$$\Phi(t)e^{B(0)t} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \begin{pmatrix} e^t \cos t & e^t \sin t \\ -e^{-t} \sin t & e^{-t} \cos t \end{pmatrix}.$$

$\square$

Example 2.3. Consider the linear system

$$x' = \begin{pmatrix} 4t + 2 & -e^{3t^2+3t} \\ e^{-3t^2-3t} & -2t-3 \end{pmatrix} x. \tag{2.7}$$

We decompose (2.7) by

$$x' = (A(t) + B(t))x,$$

where

$$A(t) = \begin{pmatrix} 4t+1 & 0 \\ 0 & -2t-2 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 & -e^{3t^2+3t} \\ e^{-3t^2-3t} & -1 \end{pmatrix}.$$

Then, the fundamental matrix of $x' = A(t)x$ is given by

$$\Phi(t) = \begin{pmatrix} e^{2t^2+t} & 0 \\ 0 & e^{-t^2-2t} \end{pmatrix}.$$

Also, we have

$$A(t)B(t) - B(t)A(t)$$

$$= \begin{pmatrix} 4t+1 & -(4t+1)e^{3t^2+3t} \\ -(2t+2)e^{-3t^2-3t} & 2t+2 \end{pmatrix} - \begin{pmatrix} 4t+1 & (2t+2)e^{3t^2+3t} \\ (4t+1)e^{-3t^2-3t} & 2t+2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -(6t+3)e^{3t^2+3t} \\ -(6t+3)e^{-3t^2-3t} & 0 \end{pmatrix} = B'(t).$$

Therefore, by the transformation $x = \Phi(t)y,$ (2.7) is converted into

$$y' = B(0)y, \quad \text{where} \quad B(0) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Since the eigenvalues of $B(0)$ is $\lambda = 0,$ we have

$$e^{B(0)t} = E + tB(0) = \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}.$$

Hence, we have the fundamental matrix of (2.7) as

$$\Phi(t)e^{B(0)t} = \begin{pmatrix} e^{2t^2+t} & 0 \\ 0 & e^{-t^2-2t} \end{pmatrix}\begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix} = \begin{pmatrix} (1+t)e^{2t^2+t} & -te^{2t^2+t} \\ te^{-t^2-2t} & (1-t)e^{-t^2-2t} \end{pmatrix}.$$

□

Theorem 2.2. *Consider the linear system*

$$x' = (A(t) + a(t)E + b(t)C(t))x, \tag{2.8}$$

where $A(t)$ *is a continuous* $n \times n$ *matrix function,* $a(t), b(t)$ *are continuous functions, and* $C(t)$ *is a continuously differentiable* $n \times n$ *matrix function. If* $C(t)$ *satisfies*

$$C'(t) = A(t)C(t) - C(t)A(t),$$

then (2.8) *is converted into*

$$y' = (a(t)E + b(t)C(0))y,$$

under the transformation $x = \Phi(t)y$ *where* $\Phi(t)$ *is the fundamental matrix of* $x' = A(t)x$ *satisfying* $\Phi(0) = E.$

Proof. By the transformation $x = \Phi(t)y,$ the left hand side of (2.8) is given by

$$x' = (\Phi(t)y)' = A(t)\Phi(t)y + \Phi(t)y'.$$

Also, the right hand side of (2.8) is given by

$$(A(t) + a(t)E + b(t)C(t))x = (A(t) + a(t)E + b(t)C(t))\Phi(t)y.$$

Therefore (2.8) is converted into

$$y' = (a(t)E + b(t)\Phi(t)^{-1}C(t)\Phi(t))\,y.$$

Since $(\Phi(t)^{-1})' = -\Phi(t)^{-1}A(t)$, we have

$$\begin{aligned}(\Phi(t)^{-1}C(t)\Phi(t))' & \\ &= -\Phi(t)^{-1}A(t)c(t)\Phi(t) + \Phi(t)^{-1}C'(t)\Phi(t) + \Phi(t)^{-1}C(t)A(t)\Phi(t) \\ &= \Phi(t)^{-1}(-A(t)C(t) + C'(t) + C(t)A(t))\,\Phi(t) = O.\end{aligned}$$

Then, we have that $\Phi(t)^{-1}C(t)\Phi(t) \equiv C(0)$. Hence, (2.8) is converted into

$$y' = (a(t)E + b(t)C(0))\,y.$$

This completes the proof. □

Example 2.4. Consider the linear system

$$x' = \begin{pmatrix} t^2 + 2t\cos 2t & -1 + 2t\sin 2t \\ 1 + 2t\sin 2t & t^2 - 2t\cos 2t \end{pmatrix} x. \tag{29}$$

We decompose (2.9) as

$$x' = (A + a(t)E + b(t)C(t))x,$$

where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad a(t) = t^2, \quad b(t) = 2t, \text{ and } C(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{pmatrix}.$$

Then, the fundamental matrix of $x' = Ax$ is given by

$$\Phi(t) = e^{At} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Also, we have

$$\begin{aligned} AC(t) - C(t)A &= \begin{pmatrix} -\sin 2t & \cos 2t \\ \cos 2t & \sin 2t \end{pmatrix} - \begin{pmatrix} \sin 2t & -\cos 2t \\ -\cos 2t & -\sin 2t \end{pmatrix} \\ &= \begin{pmatrix} -2\sin 2t & 2\cos 2t \\ 2\cos 2t & 2\sin 2t \end{pmatrix} = C'(t). \end{aligned}$$

Therefore, by the transformation $x = \Phi(t)y$, (2.9) is converted into

$$y' = (a(t)E + b(t)C(0))y, \quad \text{where} \quad C(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\alpha(t) = \int_0^t a(s)\,ds = \frac{1}{3}t^3$ and $\beta(t) = \int_0^t b(s)\,ds = t^2$. Since the eigenvalues of $C(0)$ are $\lambda = \pm 1$, Lemma 2.2 implies that

$$e^{\alpha(t)E+\beta(t)C(0)} = \frac{1}{2}e^{\alpha(t)+\beta(t)}(C(0)+E) - \frac{1}{2}e^{\alpha(t)-\beta(t)}(C(0)-E)$$

$$= \begin{pmatrix} e^{\alpha(t)+\beta(t)} & 0 \\ 0 & e^{\alpha(t)-\beta(t)} \end{pmatrix} = \begin{pmatrix} e^{\frac{1}{3}t^3+t^2} & 0 \\ 0 & e^{\frac{1}{3}t^3-t^2} \end{pmatrix}.$$

Therefore, the fundamental matrix of (2.9) is given by

$$\Phi(t)e^{\alpha(t)E+\beta(t)C(0)} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} e^{\frac{1}{3}t^3+t^2} & 0 \\ 0 & e^{\frac{1}{3}t^3-t^2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{1}{3}t^3+t^2}\cos t & -e^{\frac{1}{3}t^3-t^2}\sin t \\ e^{\frac{1}{3}t^3+t^2}\sin t & e^{\frac{1}{3}t^3-t^2}\cos t \end{pmatrix}.$$

□

Example 2.5. Consider the linear system

$$x' = \begin{pmatrix} \cos t & t\cos t + t\sin t & t+1 \\ t\cos t + t\sin t & \cos t & t\cos t - t\sin t \\ -t-1 & t\cos t - t\sin t & \cos t \end{pmatrix} x. \qquad 2.10)$$

We decompose (2.10) as

$$x' = (A + a(t)E + b(t)C(t))x,$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad a(t) = \cos t, \quad b(t) = t, \text{ and}$$

$$C(t) = \begin{pmatrix} 0 & \cos t + \sin t & 1 \\ \cos t + \sin t & 0 & \cos t - \sin t \\ -1 & \cos t - \sin t & 0 \end{pmatrix}.$$

Then, the fundamental matrix of $x' = Ax$ is given by

$$\Phi(t) = e^{At} = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}.$$

Also, we have

$$AC(t) - C(t)A = \begin{pmatrix} -1 & \cos t - \sin t & 0 \\ 0 & 0 & 0 \\ 0 & -\cos t - \sin t & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ \sin t - \cos t & 0 & \cos t + \sin t \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cos t - \sin t & 0 \\ \cos t - \sin t & 0 & -\cos t - \sin t \\ 0 & -\cos t - \sin t & 0 \end{pmatrix} = C'(t).$$

Therefore, by the transformation $x = \Phi(t)y$, (2.10) is converted into

$$y' = (a(t)E + b(t)C(0))y, \quad \text{where} \quad C(0) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of $C(0)$ is given by $|\lambda E - C(0)| = \lambda(\lambda - 1)$ $(\lambda + 1)$. Thus, the eigenvalues of $C(0)$ are $\lambda = 0, \pm 1$. Since

$$\frac{1}{|\lambda E - C(0)|} = -\frac{1}{\lambda} + \frac{1}{2(\lambda - 1)} + \frac{1}{2(\lambda + 1)},$$

we define the projections on the eigenspaces as follows:

$$P_1 = -(C(0) - E)(C(0) + E) = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix},$$

$$P_2 = \frac{1}{2} C(0)(C(0) + E) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_3 = \frac{1}{2}C(0)(C(0)-E) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Let $\alpha(t) = \int_0^t a(s)\,ds = \sin t$ and $\beta(t) = \int_0^t b(s)\,ds = \frac{1}{2}t^2$. Then, we have

$$e^{\alpha(t)E+\beta(t)C(0)} = e^{\alpha(t)}P_1 + e^{\alpha(t)+\beta(t)}P_2 + e^{\alpha(t)-\beta(t)}P_3$$

$$= e^{\sin t}\begin{pmatrix} 1 & -1+e^{\frac{1}{2}t^2} & -1+e^{\frac{1}{2}t^2} \\ 1-e^{-\frac{1}{2}t^2} & -1+e^{-\frac{1}{2}t^2}+e^{\frac{1}{2}t^2} & -1+e^{\frac{1}{2}t^2} \\ -1+e^{-\frac{1}{2}t^2} & 1-e^{-\frac{1}{2}t^2} & 1 \end{pmatrix}.$$

Consequently, we have the fundamental matrix of (2.10) as

$$\Phi(t)e^{\alpha(t)E+\beta(t)C(0)}$$

$$= e^{\sin t}\begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}\begin{pmatrix} 1 & -1+e^{\frac{1}{2}t^2} & -1+e^{\frac{1}{2}t^2} \\ 1-e^{-\frac{1}{2}t^2} & -1+e^{\frac{1}{2}t^2}+e^{-\frac{1}{2}t^2} & -1+e^{\frac{1}{2}t^2} \\ -1+e^{-\frac{1}{2}t^2} & 1-e^{-\frac{1}{2}t^2} & 1 \end{pmatrix}.$$

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